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# Computing the Brouwer Degree in $R^2$

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**Abstract.** A very simple rigorous procedure is derived for computing the Brouwer degree in $R^2$, a generalization of the zero-counting integral $\oint f'(z)\,dz/f(z)$, for functions which are Lipschitz continuous on a piecewise linear path of integration, using only computed or observed values of $f(z)$, a bound for the error in them, and a bound for the Lipschitz constant. It is used to locate zeros and to test the numerical significance of zeros found by other methods.

1. Introduction. Let $\gamma$: $[a, b] \to R^2$ be any positively oriented Jordan curve in the plane, and let $f$: $R^2 \to R^2$ be continuous. The well-known *degree of $f$ on $\gamma$*, denoted by $d(f, \gamma)$, is the Brouwer degree of $f(z)/||f(z)||$ restricted to $\gamma$, or the winding number of $f(\gamma(t))$ with respect to the origin. It is defined if there are no zeros of $f$ on $\gamma$; it is an additive function of the path $\gamma$; and it is invariant under continuous deformations of $f$ or $\gamma$ which do not put zeros of $f$ on $\gamma$.

If $f$ is analytic, of course, $d(f, \gamma)$ is the sum of the multiplicities of the zeros of $f$ inside $\gamma$. Of greater interest, however, is the fact that if $d(f, \gamma) \neq 0$, then there must be at least one zero of $f$ inside $\gamma$, *even if $f$ is not analytic.* (Otherwise, $\gamma$ could be continuously deformed to a point without crossing a zero of $f$, so $d(f, \gamma) = 0$.) This fact can be used to prove the existence of zeros of functions about which very little is known and to map out areas in which their zeros must lie.

An isolated zero of a nonanalytic $f$ such that $d(f, \gamma) = 0$ for every sufficiently small $\gamma$ containing it (e.g., the zero of $f(z) = |z|$) cannot be detected this way, but it is not numerically significant, for it can be removed by an arbitrarily small perturbation of $f$. On the other hand, if $d(f, \gamma) \neq 0$, there is at least one zero of $f$ inside $\gamma$ which cannot be removed this way. Therefore, a zero which has been found by any other method can be tested for numerical significance (in this sense) by putting a small curve $\gamma$ around it and computing $d(f, \gamma)$.

2. Computing the Degree. Delves and Lyness [1] have given algorithms, based on quadrature formulas, for computing $d(f, \gamma)$ when $f$ is analytic and presumed to be known exactly. These algorithms are superior to ours when $f$ is sufficiently well-behaved. However, our algorithm also works for nonanalytic $f$ and for functions known only approximately. It is completely rigorous and does not depend on error bounds for quadrature formulas which may be hard to compute. The computer program for our algorithm is extraordinarily simple, thereby reducing the amount of computing time used for "housekeeping" activities.

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We shall identify $R^2$ with the complex plane $C$, but the structure of $C$ will be used only for convenience in notation. We shall use the norm $||z|| = \max(|\text{Re } z|, |\text{Im } z|)$ because it is easy to compute; but other norms could also be used, with necessary changes.

We assume that there is a constant $L$ such that $||f(z_1) - f(z_2)|| < L||z_1 - z_2||$ for all $z_1, z_2 \in \gamma$, and that the computed or observed value $f^*(z)$ obeys $||f^*(z) - f(z)|| \leqq \epsilon$ for all $z \in \gamma$, where $\epsilon$ includes the effects of round-off, interpolation and observation errors.

The degree cannot be found by this method unless $||f^*(z)|| - 2\epsilon \geqq \mu > 0$ for all values of $f^*(z)$ computed in the algorithm. Accordingly, the algorithm will give an abnormal termination whenever this condition fails.

We choose a subdivision $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ so that, for $k = 0, 1, \cdots, n - 1$, both $F_{k+1}$ and $f(\gamma(t))$ for $t_k < t < t_{k+1}$ lie in the open square $S_k = \{s \mid ||s - F_k|| < ||F_k||\}$, where we define $F_i = f^*(\gamma(t_i))$.

If $\gamma$ is piecewise linear (i.e., a polygon appropriately parameterized), then this is most easily done by setting

$$t_{k+1} = \min\left[b_k, t_k + \frac{||F_k|| - 2\epsilon}{L||\gamma(b_k) - \gamma(a_k)||}(b_k - a_k)\right],$$

where $a_k \leqq t_k < b_k$ and $\gamma$ is linear on $[a_k, b_k]$. (Any other $\gamma$ can be approximated arbitrarily well by a piecewise linear function.)

Since $F_k$, $F_{k+1}$ and $f(\gamma(t))$ for $t_k \leqq t \leqq t_{k+1}$ are all in the convex set $S_k$, which does not contain the origin, the winding number of $f(\gamma(t))$ is not changed if we perturb each piece of it, in the obvious way, to the straight line segment from $F_k$ to $F_{k+1}$.

Hence, $d(f, \gamma)$ is the winding number of the polygon with vertices $F_k$, $k = 0, 1, \cdots, n$, and all we need to do is to count the number of times this polygon crosses the positive real axis, and in which directions. The counting can be combined with the determination of the $t_k$.

If $\text{Re } F_k \leqq |\text{Im } F_k|$, then $S_k$ does not contain any of the positive real axis, so we can exclude the segment from $F_k$ to $F_{k+1}$ from the counting.

If $\text{Re } F_k > |\text{Im } F_k|$, then $S_k$ does not contain any of the nonpositive real axis. If $\text{Im } F_k$ and $\text{Im } F_{k+1}$ have opposite signs, then the segment from $F_k$ to $F_{k+1}$ crosses the positive real axis once in the direction determined by the sign of $\text{Im } F_{k+1} - \text{Im } F_k$. If they have the same sign, the segment does not cross. Moreover, if 0 is consistently treated as a negative number in these tests, they are exhaustive, and the winding number is unaffected, since this treatment corresponds to lowering the entire polygon an arbitrarily small distance.

Since each $F_k$ belongs to two segments, its sign is tested twice. It is essential that the two tests give the same result. In particular, $F_0$ and $F_n$ must be equal, and they should not be calculated separately, lest a round-off error produce different values for the signs of their imaginary parts.

The following Algol 60 procedure performs the part of the algorithm associated with the part of $\gamma$ which is a line segment from $u$ to $v$, reparameterized with $0 \leqq t \leqq 1$. The value of the procedure is the crossing count. The departures from Algol syntax, which we use for clarity and brevity, have obvious meanings. The function norm($z$) is the previously defined norm$||z||$.

```
integer procedure Degree(u,v,abnormal termination);
    complex u,v; label abnormal termination;
begin
    real t; integer d; complex Fk,Fk1;
    comment f, ε, L, μ and norm are global identifiers described in the text;
    integer procedure SIP(z); complex z;
        if Im(z)>0 then SIP:=1 else SIP:=0;
    d:=0; t:=0; Fk:=f(u);
    comment Statements may be placed here to compute special values
    of ε, L and μ applicable to this segment only;
    for t:=t+(norm(Fk)-2×ε)/(L×norm(v-u)) while t < 1 do
    begin
        if norm (Fk)-2×ε<μ then go to abnormal termination;
        Fk1:=f((1-t)×u+t×v);
        if Re(Fk)>abs(Im(Fk)) then d:=d+SIP(Fk1)-SIP(Fk);
        Fk:=Fk1;
    end;
    if Re(Fk)>abs(Im(Fk)) then Degree:=d+SIP(f(v))-SIP(Fk) else Degree:=d;
end
```

Then, for a polygon $\gamma$ with vertices $z_1, z_2, \cdots, z_m$, $d(f, \gamma)$ = Degree($z_1$, $z_2$, label) + Degree($z_2$, $z_3$, label) + $\cdots$ + Degree($z_m$, $z_1$, label). Notice that the procedure and calls are constructed so that the two evaluations of each $F_k$ give the same result.

If a procedure of this type is used, $\epsilon$, $L$ and $\mu$ need only apply to the particular segment involved, and may therefore be recomputed inside each procedure call. The values of $L$ and $\epsilon$ should be slightly larger than they theoretically need to be so that a round-off error in the procedure computations will not permit $F_k$, $F_{k+1}$ or $f(\gamma(t))$ for any $t_k \leqq t \leqq t_{k+1}$ to lie outside $S_k$.

The value of $\mu$, however, is a matter of discretion; any positive value will suffice. (If $\mu \leqq 0$, the algorithm might never terminate.) The number of iterations in the *for* loop does not exceed $L||u - v||/\mu$. The smaller $\mu$ is, the longer the procedure may work before giving an abnormal termination.

**3. Finding Zeros.** If $d(f, \gamma) \neq 0$ and we divide the interior of $\gamma$ into two regions enclosed by $\gamma_1$ and $\gamma_2$, then, since $d(f, \gamma) = d(f, \gamma_1) + d(f, \gamma_2)$, either $d(f, \gamma_1) \neq 0$ or $d(f, \gamma_2) \neq 0$ or both (provided they can be computed), so we have a smaller region or regions in which zeros of $f$ must lie. By repeating this process and discarding curves on which the degree is 0, we can locate some of the zeros of $f$ inside the original curve, or all of them if $f$ is analytic. This is actually a two-dimensional analog of binary search.

If $f$ is not analytic, then all numerically insignificant zeros will be overlooked, and some significant zeros *might* be overlooked if they lie inside a curve on which the degree is 0. It may be wise, therefore, not to discard any curves, at least in the first few steps. If $f$ has nonisolated zeros, it might be impossible to divide a region so that the degree can be computed on the two parts.

We shall use triangular regions for ease in programming, although rectangular regions might give more readable results for some purposes.

Suppose the degree of $f$ is nonzero on the positively oriented triangle whose

vertices are $(u, v, w)$, respectively. We split the triangle along the median from $u$ to $s = (v + w)/2$. Then we find the degree of $f$ on the triangle $(s, u, v)$, if possible, and the degree on the triangle $(s, w, u)$ by subtraction. If one of the zeros of $f$ is too close to the median, we may have to move $s$ a bit. If the zeros of $f$ are not isolated, it may be impossible to split the triangle in this way, in which case we stop after a predetermined number $\Sigma$ of attempts.

Then, we repeat the process on the triangles $(s, u, v)$ and $(s, w, u)$, or on one of them if the degree of $f$ on the other is 0. *The order of the vertices is important.* Both triangles are positively oriented and both begin with the new vertex $s$. Consistent use of these rules guarantees that alternate "generations" of triangles are similar, except for the first "generation". Since their areas are halved at each step, their diameters tend to 0. Their shape may be altered somewhat if $s$ cannot be chosen equal to $(v + w)/2$, but we will still get convergence if $s = \theta v + (1 - \theta)w$, where $0 < \alpha \leqq \theta \leqq \beta < 1$.

The following Algol 60 procedure (with some departures from Algol syntax) finds zeros to an accuracy $\rho$, if possible, where *duvw* is the degree of $f$ on the original triangle $(u, v, w)$, $\alpha = .5$ and $\beta = .75$. The procedure is not the most efficient—for example, some line segments are processed twice by the procedure Degree—but the necessary improvements are obvious.

```
procedure Find zeros(u,v,w,duvw); complex u,v,w; integer duvw;
if duvw≠0 then
begin
    real θ; complex s; integer dsuv;
    comment ρ, Σ, Degree and norm are global identifiers described in the text;
    if max(norm(u−v),norm(u−w),norm(v−w))>ρ then
    for θ:= .5 step .25/(Σ−1) until .75 do
    begin
        s:=θ×v+(1−θ)×w;
        dsuv:=Degree(s,u,fail)+Degree(u,v,fail)+Degree(v,s,fail);
        Find zeros (s,u,v,dsuv);
        Find zeros (s,w,u,duvw−dsuv);
        go to exit;
fail:
    end;
    print (duvw,u,v,w);
exit:
end
```

In a $\delta$-neighborhood of a $k$-fold zero of an analytic function, or a similar zero of a nonanalytic function, the average value of $||F_k|| - 2\epsilon$ can usually be taken to be $O(\delta^k)$ as $\delta \to 0$. If the value of $L$ is constant, the number of function evaluations required to split a triangle in the neighborhood will be $O(\delta^{1-k})$, and it will be bounded for simple zeros. For the zeros of $z^2 + |z|$, for example, this bound was typically between 10 and 20. Apart from its other advantages, the algorithm is a practical way to compute well-conditioned zeros of nonanalytic functions.

It is not practical to compute multiple or very close zeros to arbitrary accuracies unless the value of $L$ can be suitably revised. For example, if $f(z)$ is a polynomial, we can expand $f'(z) = \sum q_i(z - c)^i$ about the center $c$ of the line segment. This

can be done by synthetic division with tolerable round-off errors [4]. If $\lambda$ is the length of the segment, then $L = 2^{1/2} \sum |q_i|(\lambda/2)^i$ is $O(\delta^{k-1})$ when the segment is in a $\delta$-neighborhood of a $k$-fold zero. For polynomials, however, other methods [3] of finding all the zeros are probably better.

**4. Numerical Examples.** The procedures were first used to locate the zeros 0, $-1$ and $\frac{1}{2} \pm \frac{1}{2}\sqrt{3}i$ of the simple nonanalytic function $f(z) = z^2 + \bar{z}$. Since the zeros all obey $|z| \leqq 1$ (a fact that could have been deduced easily from the expression for $f(z)$), the triangle with vertices at $\pm 2 - i$ and $2i$ contains all of them. An appropriate Lipschitz constant for $z^2$ on this triangle is the maximum of $|\mathrm{Re}(d/dz)z^2| + |\mathrm{Im}(d/dz)z^2|$, or 6. We add the obvious Lipschitz constant 1 for $\bar{z}$ to obtain $L = 7$. The other parameters were taken as $\epsilon = 10^{-8}$, $\mu = 10^{-8}$, $\Sigma = 6$ and $\rho = .05$.

A nonrecursive variant of the procedure "Find zeros" was used in which the degree along the segment $uv$ was retrieved from previous calculations instead of being calculated anew. Also, no triangles were discarded from the first five generations. This was necessary because the degree of the zero at $z = 0$ is $-1$ and the other three have degree 1.

The roots were separated in the fifth generation and had been found to the desired accuracy by the 15th. The number of times $f(z)$ was evaluated, which is a good measure of the quantity of computation, was 27 times per splitting, on the average. The average would have been smaller if the value of $L$ had been suitably revised.

Similar procedures were used on the polynomials $z^{10} + 1$ and $z^5(z + 1)$, except that triangles with zero degree were discarded immediately. The Lipschitz constant $L$ was obtained by the aforementioned synthetic division process.

The number of function evaluations required to *separate* the zeros was large—about 12,000 and 4,700, respectively—and the number required to split a triangle containing multiple zeros was also large. However, the average number required to split a small triangle containing a simple zero was only about 12.

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